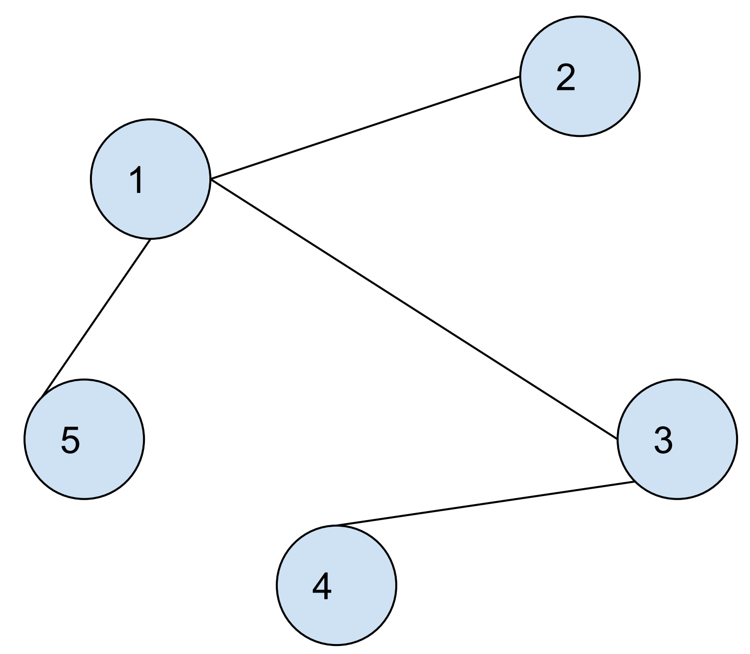
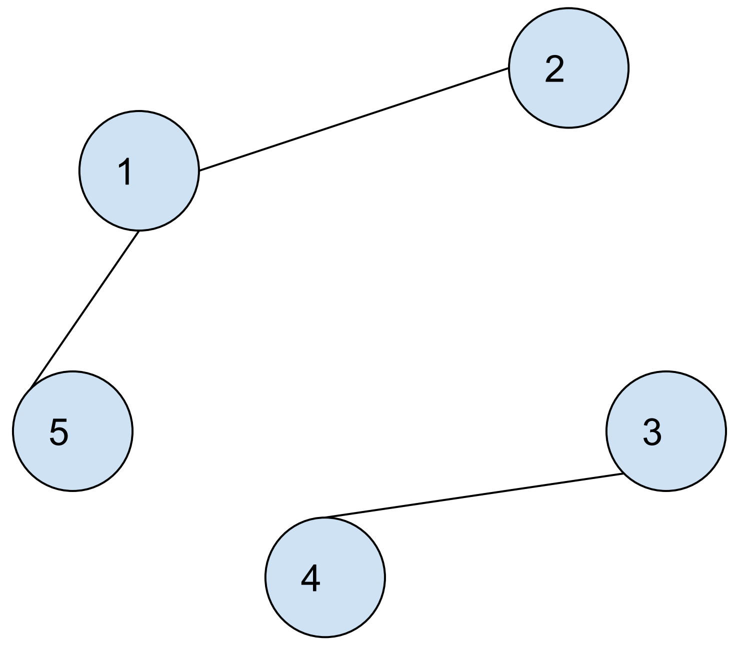
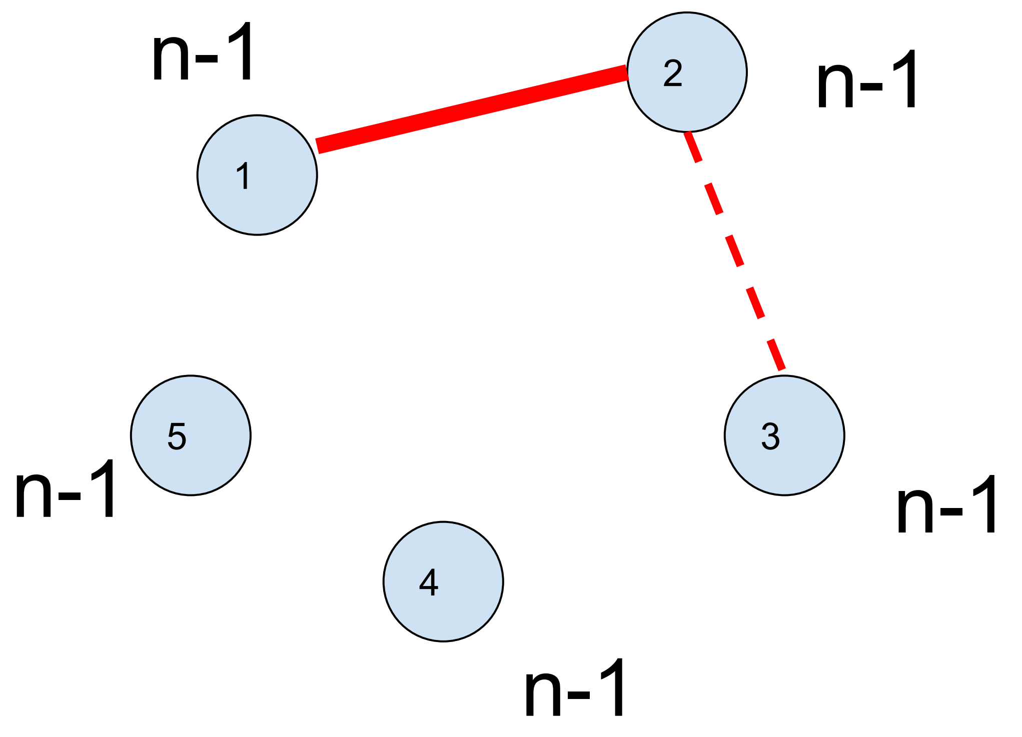
Exercise 6 [5 points] Use the inclusion-exclusion principle to determine the number of graphs with no vertices of degree 0 on the n-element vertex set $V = \{1, 2, \ldots, n\}$ as a function of n. Warning: The resulting formula is a sum and not a 'nice' formula like a Binomial coefficient. Be sure to define the sets you are using, what their sizes are and how the answer to the question using the inclusion-exclusion principle. *Tip: In how many graphs on V does a given vertex $i \in V$ have degree 0?*





(n-1)+(n-2)+(n-3)+...+1
=(1+(n-1))*(n-1)/2=n(n-1)/2
макс количество ребер в графе из n вершин

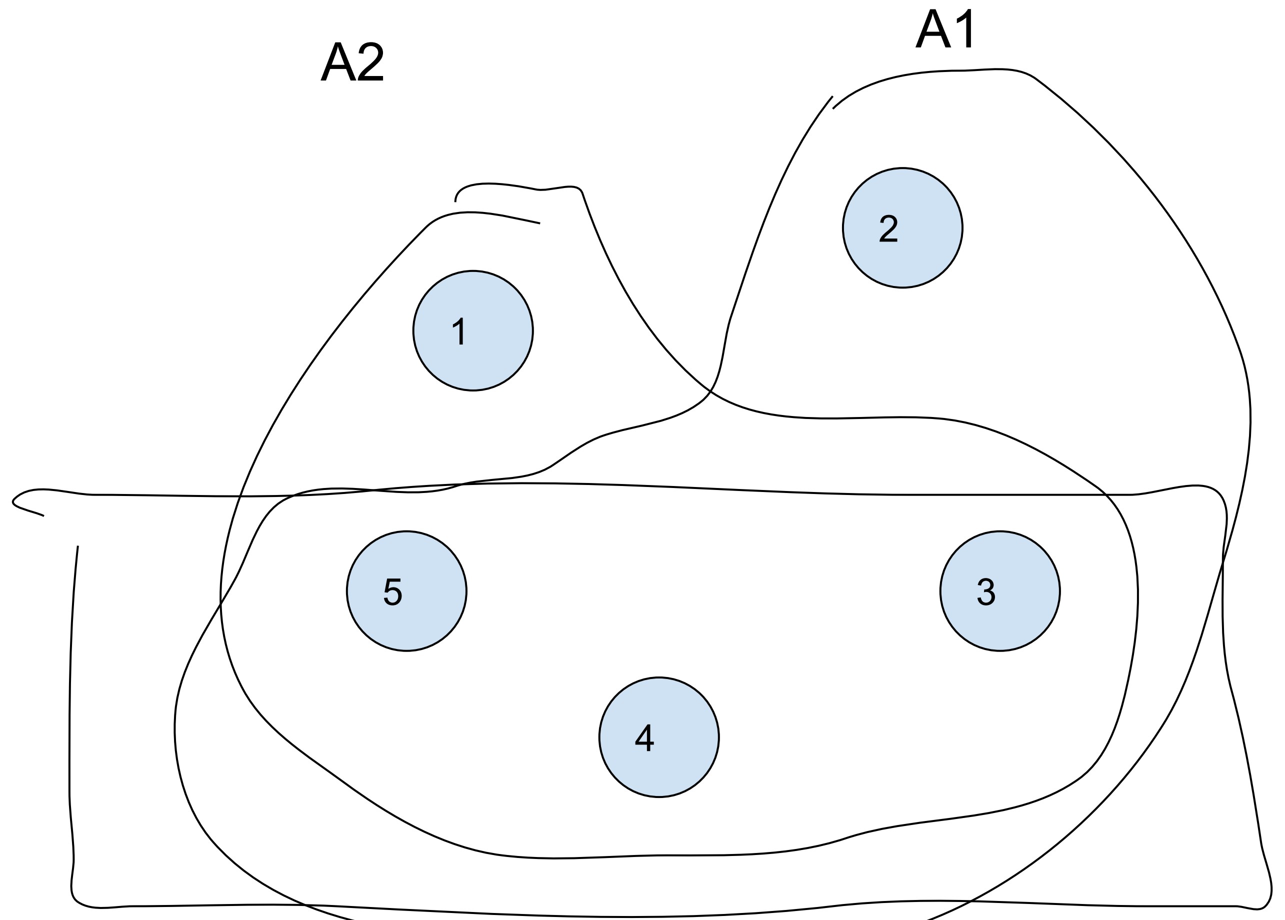


(n-1)n/2
макс
количество
ребер в
графе из n
вершин

2^[(n-1)n/2]
количество
всевозможных
графов из n
вершин

множество графов

$$\binom{n}{2} = \frac{n!}{(n-2)!\,2!}$$



A1 ^ A2

|A1 ^ A2|=2^(C(n-2,2))

Determine the number of graphs with no vertices of degree o on a given n-element vertex set V.

The total number of simple graphs with n vertices is $2^{\binom{n}{2}}$. We want to find the number of graphs with degree o and subtract it from the total.

Let v be an arbitrary vertex in V. Let's denote the set

$$A_v = \{E \in 2^{\binom{n}{2}} : deg(v) = 0\}$$

It is the set of edge sets (which correspond to graphs) for which the degree of v is o. Let's calculate the size of A_v: We discard all edge sets (graphs) in which there are edges that are incident to v. For the rest of the $n-1$ vertices we do not care if there any incident or not, thus the total number of such graphs is

$$|A_v| = 2^{\binom{n-1}{2}}$$

Now consider $|A_u \cap A_v|$, such that $u \neq v$. We discard edges sets (graphs) in which there are edges adjacent to u or to v or both (if there is an edge u, v).

$$|A_v \cap A_u| = 2^{\binom{n-2}{2}}$$

More generally, if there are $|U| = k$ such vertices with degree zero, the total number of graphs is

$$|\bigcap_{u \in U}| = 2^{\binom{n-k}{2}}$$

Now, we can apply the inclusion-exclusion formula

$$|\bigcup_{i=1}^{n} A_i| = \sum_{\emptyset \neq U \subseteq \{1,2,\ldots,n\}} (-1)^{|U|-1} |\bigcap_{u \in U} A_u|$$

$$|\bigcup_{i=1}^{n} A_i| = \sum_{\emptyset \neq U \subseteq \{1,2,\ldots,n\}} (-1)^{k-1} 2^{\binom{n-k}{2}}$$

So, the total number is $2^{\binom{n}{2}}$ minus $|\bigcup_{i=1}^{n} A_i|$.

Is this line of reasoning correct?